Two-Body Mass-Shell Constraints in a Constant Magnetic Field (Neutral Case)

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A constant homogeneous magnetic field is applied to a composite system made up of two scalar particles with opposite charges. Motion is described by a pair of coupled Klein–Gordon equations that are written in closed form with help of a suitable representation. The relativistic symmetry associated with the magnetic field is carefully respected. Considering eigenstates of the pseudomomentum four-vector, we separate out collective variables and obtain a three-dimensional reduced equation, posing a nonconventional eigenvalue problem. The velocity of the system as a whole (with respect to the frames where the field is purely magnetic) generates "motional terms" in the formulas; these terms are taken into account within a manifestly covariant framework.

1. INTRODUCTION

The theory of many-particle systems in external fields requires particular caution, even in the simple framework of nonrelativistic mechanics: as soon as all the constituent masses are of comparable magnitude, it becomes difficult to disentangle the dynamics of relative variables from the motion of the center of mass.

The case of a globally neutral system of charges imbedded in a constant homogeneous magnetic field is of special interest, however, because (under very general assumptions) it enjoys this property that the total *pseudomomentum* $\vec{C} = \Sigma \vec{p} + e\vec{A}$ is conserved and has mutually commuting components [1–4]. This exceptional circumstance enables us to separate, in a generalized sense, relative motion, and therefore provides a clear-cut definition of what is the spectrum of the system [4]. Relativistic corrections have been considered [3] in a three-dimensional framework; this is certainly suffi-

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cient in a large number of applications, but fails to account for the relativistic symmetry. Indeed the constant magnetic field has the peculiarity that it does not correspond to a unique "laboratory frame." When a constant homogeneous electromagnetic field is seen as purely magnetic in some frame (conventionally referred to as the lab frame), such a frame cannot be unique [5], thus total energy, if defined as the (conserved) time component of linear momentum, is affected by this ambiguity. All the directions eligible for the time axis of a possible lab frame span a two-dimensional plane (E_L) with hyperbolic metric; so we are led to pay attention to special Lorentz transformations in this "longitudinal plane." Thus a four-dimensional spacetime approach is warranted in order to keep under control the full relativistic symmetry of motion.

In this paper we focus on two-body systems because the covariant methods of relativistic particle dynamics are well understood and more tractable in this case. In previous works [5, 6] we have indicated how the mass-shell constraints for two scalar particles undergoing mutual interaction can be minimally coupled (in closed form and remaining compatible) with an external electromagnetic field $F_{\mu\nu}$ which can be either pure electric or pure magnetic. In both cases a four-vector C^α called pseudomomentum is conserved and for *neutral systems* its four components commute among themselves. Writting down explicit equations of motion requires that we go to a new representation, adapted to the symmetries of the external field.

When, as we assume here, $F_{\mu\nu}$ is purely magnetic,[2] a further change of representation eliminates not only the collective variables conjugate to pseudomomentum, but also a fifth variable which is nothing but relative time. The outcome is a manifestly covariant equation to be solved for a reduced wave function which depends only on three spacelike degrees of freedom. The material that we published so far [5] was limited to the general lines of this approach.

In this article we explicitly carry out the change of representation and write down the reduced wave equation in a tractable form, showing the details of the various contributions it contains. In addition, we discuss whether the reduced wave equation can be considered as an eigenvalue problem, and for which parameter. We prepare an eventual perturbation theory which will ultimately result in a covariant framework for the spectroscopy of two-body systems.

In Section 2 we display the notation used and we recall several results from previous work. Section 3 is devoted to the explicit reduction of the number of degrees of freedom, and to a qualitative discussion about the

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[2] We say that a field is pure electric (resp. pure magnetic) when there exist frames where the field appears to be so; of course, these two situations exclude one another.

various terms arising in the reduced wave equation. Section 4 contains concluding remarks and an outlook.

2. BASIC EQUATIONS, SYMMETRIES

When pair creation can be neglected, a system of two scalar particles can be described by a pair of coupled Klein–Gordon equations

$$2H_a\Psi = m_a^2\Psi, \qquad a, b = 1, 2$$

referred to as the *mass-shell constraints*. Here Ψ has two arguments q_1, q_2 running in spacetime. We cover all cases of practical interest assuming that $H_a = K_a + V$. In the above formula $2K_a = (p_a - e_a A(a))^2$ is the squared-mass operator for particle a alone in the magnetic field, and V is a suitable modification of the term $V^{(0)}$ which would describe the mutual interaction in the absence of an external field; this modification is necessary in order to keep the mass-shell constraints mutually compatible when the field $F_{\alpha\beta}$ is applied.

For all vectors ξ, η we write $\xi \cdot F \cdot \eta$ for $\xi^\alpha F_{\alpha\beta}\eta^\beta$. With a similar convention $A(a) = 1/2\, q_a \cdot F$ in a Lorentz-covariant gauge.

Notice that $\xi \cdot \eta_L = \xi_L \cdot \eta_L$ and $\xi \cdot \eta_T = \xi_T \cdot \eta_T$.

An important technical point is that applying a constant magnetic field provides a unique and invariant decomposition of any four-vector ξ into longitudinal and transverse parts, say $\xi = \xi_L + \xi_T$. The orthocomplement of (E_L) in the space of four-vectors is a two-dimensional plane (E_T) endowed with elliptic metric. In any adapted frame, ξ_L (resp. ξ_T) has nonvanishing coordinates ξ^0, ξ^3 (resp. ξ^1, ξ^2).

The theory of relativistic two-body systems, formulated many years ago along the lines of "predictive mechnics" and "constraints theory" [7–9],[3] has been more recently extended to cases where some external field is present [5, 11]. Here we assume that a constant homogeneous magnetic field is applied to a pair of opposite charges, say $e_1 = -e_2 = e$.

The constraint approach employed here has several advantages over the Bethe–Salpeter equation; for example, in the particular case of isolated systems (no field applied) the dependence on relative time gets automatically factorized out [12].

It is convenient to rearrange the canonical variables as follows:

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[3] Coupled wave equations have been considered also in Refs. 10.

$$z = q_1 - q_2 \qquad Q = \frac{1}{2}(q_1 + q_2)$$

$$y = \frac{1}{2}(p_1 - p_2) \qquad P = p_1 + p_2$$

so $[z, y] = [Q, P] = -i\delta$, etc.

The Lie algebra of the Lorentz group is generated by the tensor

$$M = M_1 + M_2 = Q \wedge P + z \wedge y$$

with $M_1 = q_1 \wedge p_1$, $M_2 = q_2 \wedge p_2$. In any adapted frame, rotations in (E_T) are generated by M_{12} and boosts in (E_L) by M_{03}.

An essential ingredient of mutual interactions [7] is the quantity $\tilde{z}^2 = z^2 - (z \cdot P)^2/P^2$. But in order to avoid denominators in calculations, it is convenient to employ

$$Z = z^2 P^2 - (z \cdot P)^2 \tag{1}$$

We shall assume that

$$V^{(0)} = f(Z, P^2, y \cdot P) \tag{2}$$

This form is general enough to accommodate a large class of interactions.

Definition. When speaking of *energy-dependent interactions*, we refer to the *total* energy of isolated systems, namely $\sqrt{P^2}$.

Although Z is more practical for calculations, it would be more natural to take $\tilde{z}^2$ and P^2 as independent dynamical variables, defining $g(Z/P^2, P^2, y \cdot P) = f(Z, P^2, y \cdot P)$. Therefore we say that $V^{(0)}$ does *not* depend on (total) energy when the function f takes on the form $f = g(Z/P^2, y \cdot P)$.

Although f in (2) is supposed to be known, it would be a problem to determine V in closed form. In the external-field representation, which involves a new wave function Ψ' and new operators H'_a, K'_b, V', this problem is solved by making the ansatz

$$V' = f(\hat{Z}, P^2, y_L \cdot P_L) \tag{3}$$

where $\hat{Z} = Z'_{(0)} = (Z')_{F=0}$ (it turns out that $\hat{Z}$ commutes with $y_L \cdot P_L$). The explicit form of $\hat{Z}$ was calculated in ref. 5:

$$\hat{Z} = Z + 2 (P_L^2 z \cdot P - P^2 z_L \cdot P_L) L + P_T^2 P_L^2 L^2 \tag{4}$$

where the scalar L is defined as

$$L = \frac{P_L \cdot z}{(P_L)^2} \tag{5}$$

The equations of motion are compatible provided [5] that $\hat{Z}$ commutes with

$y_L \cdot P_L$. Let us transform (4) in order to render this commutation property manifest. First we split z as the sum of z_L and z_T in Z; hence,

$$Z = (z_T^2 + z_L^2)P^2 - (z_T \cdot P)^2 - (z_L \cdot P)^2 - 2(z_T \cdot P)(z_L \cdot P) \qquad (6)$$

Develop (4) and perform elementary manipulations using (6). We get

$$\hat{Z} = Z + 2 \, (z_T \cdot P)(z_L \cdot P) - (z_L \cdot P)^2 \frac{P_T^2}{P_L^2}$$

Using (6), we notice cancellation of the terms proportional to $(z_T \cdot P) \, (z_L \cdot P)$ and we can write

$$\hat{Z} = z_T^2 P^2 - (z_T \cdot P)^2 + P^2 \left(z_L^2 - \frac{(z_L \cdot P_L)^2}{P_L^2} \right) \qquad (7)$$

It is convenient to define the projector "orthogonal" to P_L, say

$$\Omega^\alpha_\beta = \delta^\alpha_\beta - \frac{P_L^\alpha P_{L\beta}}{P_L^2} \qquad (8)$$

because we can write

$$z_L^2 - \frac{(z_L \cdot P_L)^2}{P_L^2} = (\Omega z_L)^2 \qquad (9)$$

and we easily check that $(\Omega z)^\alpha$ commutes with $(y_L \cdot P_L)$. So we have

$$\hat{Z} = z_T^2 P^2 - (z_T \cdot P)^2 + (\Omega z_L)^2 P^2 \qquad (10)$$

which justifies the claim that $\hat{Z}$ commutes with $y_L{\cdot}P_L$. Here we notice that $\Omega z_T = z_T$ and

$$(\Omega z)^2 = z_T^2 + (\Omega z_L)^2 \qquad (11)$$

Thus we finally obtain

$$\hat{Z} = (\Omega z)^2 P^2 - (z_T \cdot P)^2 \qquad (12)$$

which is much more tractable than formula (4).

Mass-shell constraints can be replaced by their sum and difference, so we set

$$\mu = \frac{1}{2} (m_1^2 + m_2^2), \qquad \nu = \frac{1}{2} (m_1^2 - m_2^2)$$

The explicit form of K'_1 and K'_2 was given in ref. 5. Equations (3.36) of ref. 5 yield in the present notation[4]

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[4] To prevent confusion, we now avoid using the vector L^α introduced in ref. 5, because $L \cdot L$ was distinct from the square of the scalar function L.

$$K_1' + K_2' = K_1 + K_2 - 2T \frac{y_L \cdot P_L}{P_L^2} + \frac{T^2}{(P_L)^2} \tag{13}$$

where

$$T = K_1 - K_2 - y_L \cdot P_L \tag{14}$$

and the difference is

$$K_1' - K_2' = y_L \cdot P_L \tag{15}$$

It is noteworthy that M_{03} and M_{12} are not affected by going over to the external-field representation. In other words, we can write

$$M_{12}' = M_{12}, \qquad M_{03}' = M_{03} \tag{16}$$

Indeed the transformation from Ψ to Ψ' is formally generated by $B = LT$, where L and T are given by (5) and (14), respectively.[5] Commutation of L with M_{12} and M_{03} is obvious. For commutation of T, the only point to be checked is that $K_1 - K_2$ actually commutes with M_{12}. But $K_a = K(a)$, where $K(a)$ is the (half-squared) squared-mass operator for particle a alone in the field. We know the constants of the motion in the one-body sector [14]. In particular, we know that K_a commutes with both $(M_a)_{03}$ and $(M_a)_{12}$. Thus T commutes with M_{03} and M_{12}. Finally, B shares the same property, which formally proves (16). Let us prove the following:

Proposition. Angular momentum in (E_T) and boost in (E_L) are constants of the motion.

In other words, we claim that our squared-mass operators both commute with the transverse and longitudinal components of the total angular momentum.

Working in the external-field representation, all we need is to prove that M_{03} and M_{12} commute with both $K_a' + V'$, or equivalently, with $K_1' + K_2' + 2V'$ and with $y_L \cdot P_L$. Commutation with K_1' and K_2' separately is ensured from the properties of single-particle motion in the field. Moreover, $y_L \cdot P_L$ is invariant under any spacetime rotation. The last point to check is whether M_{03} and M_{12} actually commute with V'. It is sufficient that they commute with all arguments of f in formula (3), which is true because these three arguments are manifestly Lorentz-invariant.

For completeness, we recall here that pseudomomentum, originally represented by

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[5] For a more rigorous exposition we should start with Ψ' from the outset. See ref. 13.

$$C = P + \frac{e}{2} z \cdot F$$

keeps the same expression in the external-field representation ($C' = C$), and is also conserved [5].

2.1. Ultimate Representation

For neutral systems, a further transformation inspired by the work of Grotch and Hegstrom [3], and similar to a gauge transformation, enables you to get rid of the Q variables. Transforming the wave function yields $\Psi'' = (\exp i\Gamma)\Psi'$ with the help of the unitary transformation generated by

$$\Gamma = \frac{e}{2} (z \cdot F \cdot Q) \tag{17}$$

We set

$$\mathbb{O}^\# = \exp(i\Gamma) \, \mathbb{O} \, \exp(-i\Gamma), \qquad \mathbb{O}'' = (\mathbb{O}')^\# \quad \forall \mathbb{O} \tag{18}$$

The new equations of motion

$$(H_1'' + H_2'')\Psi'' = \mu\Psi'' \tag{19}$$

$$y_L \cdot P_L \Psi'' = \nu\Psi'' \tag{20}$$

may "look like" they are translation invariant, although *they are not*. The reason is that pseudomomentum is transformed to P^α by (18), that is, $C'' = P$. Of course, P is no longer the generator of spacetime translations. These transformations now have a generator P'' which differs from P because Γ in (17) is not translation-invariant. In the ultimate representation considered here C'' generates the relativistic analog of the so-called "twisted translations" invoked in ref. 4.

From now on *we demand that pseudomomentum be diagonal with a timelike four-vector k^α as eigenvalue*. Instead of $C^\alpha \Psi = k^\alpha \Psi$, we are using our ultimate representation and write $P^\alpha \Psi'' = k^\alpha \Psi''$. Combining this requirement with (20), we obtain

$$\Psi'' = \exp(ik \cdot Q) \, \exp\left(i\nu \, \frac{z_L \cdot k_L}{|k_L|}\right)\phi \tag{21}$$

where ϕ depends on z, but only through its projection orthogonal to k_L, and additionally depends on k and on ν as parameters. In other words $\phi = \phi(\nu, k, \varpi z)$ with the following notation.

Notation. For all four-vectors ξ, we define $\varpi\xi$ as the projection of ξ onto the 3-plane orthogonal to k_L, say $(\varpi\xi)^\alpha = \xi^\alpha - (\xi \cdot k_L) \, k_L^\alpha/k_L^2$. Similarly,

$\xi_{\perp}$ denotes the projection of ξ onto the 3-plane orthogonal to k. In general, $\varpi z \neq z_{\perp}$, but they coincide when k_T vanishes.

It is convenient to introduce here the *motional parameter* $\epsilon = |k_T|/|k_L|$. When ϵ does not vanish, a number of terms involving the contraction $k \cdot F$ arise. In fact $(k \cdot F)^{\alpha} = |k| E^{\alpha}$, where E^{α} is the electric field seen by an inertial observer moving with constant momentum k^{α} (*motional electric field*). We have the identity

$$\frac{1}{k_L^2} = \frac{1}{k^2} (1 - \epsilon^2) \tag{22}$$

Notice that k_T is linear in ϵ because we can write $k_T = \epsilon \Lambda k_L$ where the second-rank tensor Λ represents the boost from the direction of k_L to the direction of k_T (thus $\Lambda \cdot \Lambda = \delta$).

2.2. Explicit Formulas

The reduced (or internal) wave function ϕ must be determined through the "sum equation" $(H_1'' + H_2'') \Psi'' = \mu \Psi''$, simplified with help of (21).

Given the function f involved in (2), let us display $H_1'' + H_2''$ in detail. It is clear that

$$H_1'' + H_2'' = K_1'' + K_2'' + 2V'' \tag{23}$$

so we have to transform $(K_1' + K_2')$ and V' according to (18). We find that Q and z are unchanged, whereas

$$P^{\#} = P + \frac{e}{2} F \cdot z, \qquad P_L^{\#} = P_L \tag{24}$$

$$y^{\#} = y - \frac{e}{2} F \cdot Q \tag{25}$$

$$P^{\#2} = P^2 + eP \cdot F \cdot z + \frac{e^2}{4} (F \cdot z)^2 \tag{26}$$

$$(K_1 + K_2)^{\#} = \frac{P^2}{4} + y^2 - \frac{e}{2} z \cdot F \cdot P + \frac{e^2}{4} (z \cdot F)^2 \tag{27}$$

$$T^{\#} = y_T \cdot P_T - 2ez \cdot F \cdot y \tag{28}$$

Now we apply transformation (18) to (13), taking (27) and (28) into account. It gives

$$K_1'' + K_2'' = \frac{P^2}{4} + y^2 - \frac{e}{2} z \cdot F \cdot P + \frac{e^2}{4} (z \cdot F)^2 + \frac{T^{\#}}{P_L^2} (T^{\#} - 2y_L \cdot P_L) \tag{29}$$

with $T^{\#}$ given by (28). We know that $2V''$ must be added to this expression

in order to obtain $H_1'' + H_2''$. But in (17), $F^{\mu\nu}$ is purely transverse, therefore $(y_L \cdot P_L)^{\#} = y \cdot P_L$. We have by (3)

$$V'' = f(\hat{Z}^{\#}, P^{\#2}, y_L \cdot P_L) \tag{30}$$

where $P^{\#2}$ is as in (26) and we must compute $\hat{Z}^{\#}$ from (10) with help of (24). [We make the convention that $\hat{Z}^{\#} = (\hat{Z})^{\#}$ and not the reverse.]

To this end, we apply the transformation (18) to Eq. (10). A glance at (9) shows that $(\Omega z_L)^2$ is not affected by the transformation. Recall that z is unchanged; we notice that $z_T \cdot P^{\#} = z_T \cdot P$ because, F being purely transverse, $z_T \cdot F \cdot z$ identically vanishes. Thus, using (11), we obtain

$$\hat{Z}^{\#} = P^{\#2}(\Omega z)^2 - (z_T \cdot P)^2 \tag{31}$$

Now, Eqs. (23), (29), and (30) supplemented with (26) and (31) furnish the complete expression of $H_1'' + H_2''$ to be inserted into (19). At this stage we are in a position to carry out the reduction.

3. THREE-DIMENSIONAL REDUCTION

3.1. Calculations

After transformation to the ultimate representation we have obtained $C'' = P$. Calculations can be organized as follows: Whereas (20) fixes the dependence in the relative time, Eq. (21) allows us to factorize out the "center-of-mass motion," and we are left with the reduced wave function ϕ which arises in Eq. (21). Obviously, (20) implies that

$$y_L \cdot k_L \phi = \nu \phi \tag{32}$$

Thus ϕ depends on z only through its projection ϖz. It is clear that ϕ generally depends on ν and k as parmeters.

In searching for a reduced wave equation, we replace P^{α} and $y_L \cdot P_L$, respectively, by their eigenvalues k^{α} and ν in $H_1'' + H_2''$, and we divide by exponential factors. For any operator $\mathbb{O}$ it is convenient to use the following convention:

$$(\mathbb{O})_{\nu,k} = \mathbb{O}\big|_{y_L \cdot P_L = \nu, P = k} \tag{33}$$

The subscript k refers to the vector k, which finally contributes by its longitudinal piece only. In this procedure, a term like y^2 must be written as $y^2 \equiv (\Omega y)^2 + (y_L \cdot P_L)^2/P_L^2$. If we now introduce the projector ϖ orthogonal to k_L and use the identity (22), we obtain, for instance, with help of (32)

$$\left(\frac{P^2}{4} + y^2\right)_{\nu,k} = \frac{k^2}{4} + (\varpi y)^2 + \frac{\nu^2}{k_L^2} = \frac{k^2}{4} + (\varpi y)^2 + \frac{\nu^2}{k^2} - \epsilon^2 \frac{\nu^2}{k^2} \tag{34}$$

which is to be taken into account when computing $(K_1'' + K_2'')_{\nu,k}$ from (29).

According to (23), we have $(H_1'' + H_2'')_{v,k} = (K_1'' + K_2'')_{v,k} + 2(V'')_{v,k}$. Defining

$$R(v, k_L, k_T) = (K_1'' + K_2'')_{v,k} \tag{35}$$

$$W(v, k_L, k_T) = (V'')_{v,k} \tag{36}$$

Recalling (23), we can reduce Eq. (19) to

$$R\phi + 2W\phi = \mu\phi \tag{37}$$

Let us stress that μ is just a parameter fixed from the outset. As other parameters arise in (37), namely k and ϵ, the question of whether (37) can be considered as a spectral problem, *and for which eigenvalue*, is not yet settled and will be considered later, with help of Eqs. (41) and (46). See Eq. (45) below.

Since ϕ depends on z only through ϖz, it is important to realize that neither R nor W involves the operator $z_L \cdot k_L$. This will be checked below and will permit us to consider Eq. (37) as a three-dimensional problem involving operators R and W acting on functions of ϖz.

The explicit expression of R comes from (29), with help of (35). Since K_1'' and K_2'' are no more than quadratic in the field strength, let us make the convention that the superscripts (1) and (2), respectively, refer to the (homogeneous) linear and quadratic terms in the field. We start from (29), compute $K_1'' + K_2''$ to be inserted into (23), and further simplify with help of convention (33). The zeroth-order contribution to R is

$$R^{(0)} = \frac{k^2}{4} + \frac{v^2}{k_L^2} + (\varpi y)^2 + y_T \cdot k_T \frac{y_T \cdot k_T - 2v}{k_L^2} \tag{38}$$

Applying again identity (22) and setting

$$(S)_{v,k} = (\varpi y)^2 + (y_T \cdot k_T) \frac{y_T \cdot k_T - 2v}{k_L^2} - \epsilon^2 \frac{v^2}{k^2} \tag{39}$$

we can write

$$R^{(0)} = \frac{k^2}{4} + \frac{v^2}{k^2} + (S)_{v,k} \tag{40}$$

It is convenient to define

$$\lambda = \frac{k^2}{4} + \frac{v^2}{k^2} - \mu \tag{41}$$

so we can write

$$R^{(0)} = \lambda + \mu + (S)_{v,k} \tag{42}$$

The field-dependent terms in (29) provide

$$R^{(1)} = 4e(z \cdot F \cdot y)\frac{v}{k_L^2} - \frac{e}{2}z \cdot F \cdot k \tag{43}$$

$$R^{(2)} = \frac{e^2}{4}(z \cdot F)^2 + 4e^2\frac{(z \cdot F \cdot y)^2}{k_L^2} \tag{44}$$

We remember that F is purely transverse. Contractions involving F only depend on the transverse components; for instance, $F \cdot k$ is just a combination of the quantities k_T^α. It is noteworthy that only the transverse components of z, y arise in $R^{(1)}$, $R^{(2)}$, whereas $(S)_{v,k}$ depends on ϖy and y_T. As a whole, R depends only on ϖz and ϖy (recall y_T, z_T are pieces of ϖy, ϖz, respectively).

In view of (42)–(44), Eq. (37) may be finally written

$$\lambda\phi + [(S)_{v,k} + R^{(1)} + R^{(2)} + 2W]\,\phi = 0 \tag{45}$$

The square bracket in (45) is nothing but $(-N'')_{vk}$ provided, in the original representation, we introduce the conserved quantity

$$-N = \frac{1}{4}C^2 + (C^2)^{-1}(H_1 - H_2)^2 - (H_1 + H_2) \tag{46}$$

now represented by the operator

$$-N'' = \frac{P^2}{4} + \frac{(H_1'' - H_2'')^2}{P^2} - (H_1'' + H_2'')$$

and intimately related with the energy of relative motion.

The last term to be evaluated in (45) is W. In view of (36) we first have to write down the expression for V'', say (30). It follows that

$$W = f((\hat{Z}^\#)_{v,k}, (P^\#)_{v,k}^2, v) \tag{47}$$

In this formula $(P^\#)^2$ is given by (26) and $\hat{Z}^\#$ by (31). Making the substitutions $P \to k$ and $y_L \cdot P_L \to v$, hence $\Omega \to \varpi$, we obtain

$$(\hat{Z}^\#)_{v,k} = (P^\#)_{v,k}^2(\varpi z)^2 - (z_T \cdot k)^2 \tag{48}$$

$$(P^{\#2})_{v,k} = k^2 + ek \cdot F \cdot z + \frac{e^2}{4}(F \cdot z)^2 \tag{49}$$

It is clear that W does not involve the operator $z \cdot k_L$. Formulas (48) and (49) are to be inserted into (47), then the explicit form of W will come out.

It is natural to consider (45) as an equation for the eigenvalue λ. But we meet a complication because λ is not independent of k^2. In fact we can

solve (41) for k^2 and insert the result[6] into $(N'')_{\nu,k}$. As a result, (45) bears a nonlinear dependence on λ. A similar situation was pointed out by Rizov *et al.* [16] in the case of isolated systems undergoing energy-dependent interactions. In the presence of a magnetic field, however, the reduced wave equation is nonlinear in λ, even in the simple case where the mutual interaction term $V^{(10)}$ does not depend on P^2. This can be seen as follows: first, we notice that the occurence of $(\hat{Z}^\#)_{\nu,k}$ in W brings out a dependence on k^2, k_L^2. Second, we observe an inescapable dependence on k^2, k_L^2 in formulas (39), (43), and (44). We end up with a nonconventional spectral problem which requires special treatment, reserved for future work.

3.2. Discussion

Finally, the mass-shell constraints have been reduced to the three-dimensional problem of solving (45). This formula is nonlinear in the field strength and might be applied to strong fields.[7] Let us review the various contributions it contains. We distinguish motional terms, depending on ϵ or depending on k_T, where we know that k_T is linear in ϵ.

Loosely speaking, we could say that, inasmuch as the shape of W departs from the original form assumed by $V^{(0)}$, every thing goes as if the mutual interaction were somehow "modified by the presence of magnetic field."

3.2.1. System at Rest

The particular case where pseudomomentum is purely longitudinal (say $k_T = 0$) enjoys a particular simplicity. If we assume for a moment that k coincides with k_L, it is possible to find a frame where $\vec{k}$ vanishes whereas the electromagnetic field is purely magnetic. We refer to this situation as the case *at rest*.

In this case, $\varpi z = z_\perp$, $\varpi y = y_\perp$, and $(S)_{\nu,k}$ simply reduces to $y_\perp^2$, since k_L coincides with k.

As $z_T \cdot k$ in (48) vanishes, we notice that $(\hat{Z}^\#/P^{\#2})_{\nu,k}$ reduces to $z_\perp^2$. According to (47) and to a notation defined in Section 2, we can write

$$W = g(z_\perp^2, (P^{\#2})_{\nu,k}, \nu)$$

where $k \cdot F \cdot z$ vanishes in (49), so $(P^{\#2})_{\nu,k}$ reduces to $k^2 + \frac{1}{4}e^2(F \cdot z)^2$. If the mutual interaction does not depend on the energy, we end up with

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[6] In the case of isolated systems, there are arguments (e.g. ref. 15) for demanding that both $P \cdot p_1$ and $P \cdot p_2$ have positive eigenvalues, which amounts to requiring that $k^2/2 > |\nu|$. Provided we follow this last condition also in the presence an of external field, one root only remains admissible.

[7] The rest case in strong fields has been considered by Koller *et al.* [17]. Motional effects have been considered in the noncovariant literature, see ref. 18 and references therein.

$W = g(z_\perp^2, \nu)$. Thus, for energy-independent interactions, namely $V^{(0)} = g(Z/P^2, y \cdot P)$, W assumes the form $g(z_\perp^2, \nu)$. In other words:

At rest, the magnetic field does not modify the mutual interaction, provided this interaction is not energy-dependent.

In contrast, if $\partial V^{(0)}/\partial P^2$ does not vanish, the shape of W may substantially depart from that of $V^{(0)}$ in strong fields, owing to the contribution of $(F \cdot z)^2$ in $(P^{\#2})_{\nu,k}$. This correction to $V^{(0)}$ is a genuine "three-body" term in this sense that it vanishes if either the mutual interaction or the magnetic field is turned off (pretending that the field is generated by a ficticious "third body" located at infinity).

Looking again at Eq. (45), we see that, at rest, all surviving terms not included in W can easily be identified as covariant generalizations of the usual terms present in the nonrelativistic theory [4, 19] except for a piece of $R^{(2)}$ which depends on the relative angular momentum; see contribution of $z \cdot F \cdot y/k_L^2$ in formula (44). This contribution remains small for heavy systems ($k^2 >> F$), but might be significant for light systems ($k^2 << F$) in a strong magnetic field.

At *first order in the field strength*, however, the relative motion admits no correction other than a term proportional to ν (indeed $F \cdot k$ vanishes). For equal masses this term is zero and there is no departure from the motion of an isolated system.

3.2.2. Motional Case

When k_T is nonzero, we recognize the motional electric field contained in $z \cdot F \cdot k$. For energy-dependent potentials, and even in a weak field, this term contributes to W through (47). But of course, it may be neglected in the case of slow motion in a weak field, where both ϵ and F are considered as first-order quantities, which entails that $F \cdot k$ is a second-order quantity. On one hand, this can be seen as a stability property of the neutral two-body system under application of a constant field. But on the other hand, it forces one to go beyond the weak-field, slow-motion approximation if one wishes to compute significant corrections to the energy associated with relative motion.

4. CONCLUSION

The coupled Klein–Gordon equations describing a globally neutral system have been reduced to a three-dimensional equation involving truly motional terms and recoil effects in a covariant fashion. In this formulation the particular symmetry associated with a constant magnetic field *in space-time* is manifestly respected. After separation of the internal motion, and after factorizing the dependence on relative time, the surviving number of degrees of freedom is finally the same as in the nonrelativistic theory.

We now have a clean theoretical basis for the study of relativistic bound states in a constant magnetic field, the simplest of all the cases where an external field is present.

In the reduction procedure it was essential to consider eigenstates of pseudomomentum. The square of this vector plays the role of an effective squared mass which can be, in principle, evaluated by solving the reduced wave equation. But the eigenvalue problem involved in this equation is crucially nonconventional, for the eigenvalue arises in a nonlinear way, even if mutual interaction does not depend on the total energy. This situation requires a refinement of conventional methods; the method devised in ref. 16 will help to carry out this task in the future.

Our formulas are quadratic in the field strength and offer a starting point for investigating strong-field effects. In principle, they encompass all kinematic possibilities of the system as a whole and permit a description of ultrarelativistic situations, where $|k_T|^2 \simeq |k_L|^2$.

In the present state of the art, we notice that, in a *weak* field, the *slow* collective motion (first order in ϵ) of opposite charges interacting through a potential which does not depend on the energy escapes the above-mentioned complication, but in this case the presence of an external field results in a first-order Starck effect which obviously vanishes for generic shapes of the mutual interaction potential. For the harmonic oscillator, for instance, this remark indicates that the naive quark model enjoys some kind of stability property. But if we have perturbation theory in mind, the computation of significant corrections requires the setting of a nonconventional treatment.

In sofaras approximations are concerned, it is in order to realize that two situations are possible: Either the magnetic field is considered (as in the previous example) as a perturbation applied to the system, or, in contrast, the mutual interaction is treated as a perturbation as in the helium atom.

In the latter case, the zeroth-order approximation describes two independent particles moving in the magnetic field; in this unperturbed motion, the transverse degrees of freedom are bound by the magnetic field (corrections to the corresponding spectra are reserved for future work). We expect to avoid the pathology of "continuous dissolution" [20, 21] for two reasons: The particles we consider here have no spin, and we can impose positive individual energies, requiring that both $P \cdot p_1$ and $P \cdot p_2$ have positive eigenvalues.

The Ansatz which allows for a three-dimensional reduction in our covariant framework automatically generates various terms in the wave equation. We have seen that the importance of these terms depends on the strength of the field and on the state of motion of the system as a whole. Inspection of these terms indicates that, from a practical point of view, the shape of the mutual interaction is "somehow modified" by the magnetic field. As can be

read off from (49), the modification implied by (47) is quadratic in F and may become dominant in strong fields provided $V^{(0)}$ is energy-dependent. This point concerns most of the realistic two-body potentials. Further work is needed in order to get beyond these qualitative indications.

For the sake of simplicity, we have focused here on scalar particles, but naturally an extension to particles with spin is desirable. A generalization to globally charged systems would also be of interest.

Let us finally mention that, in principle, the contact with more conventional methods of quantum field theory could be improved, trying to derive directly all our terms from a Bethe–Salpeter equation *that takes the magnetic field into account from the start*. This would mean to remake the work of Bijtebier and Broeckaert [21] in a way which respects the particular symmetry of constant magnetic field, i.e., treating all the possible lab frames on the same footing. To our knowledge, nobody has yet carried out this task.

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